AN APPLICATION OF COUNTING IDEALS IN RAY CLASSES

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ABSTRACT. In this article, we prove a fully explicit generalized Brun-Titchmarsh theorem for an imaginary quadratic field K. More precisely, for any finite family of linearly independent linear forms with coefficients in $\mathcal{O}_{\mathbf{K}}$, we count the number of integers at which all these linear forms take prime values in $\mathcal{O}_{\mathbf{K}}$.

CONTENTS

1. INTRODUCTION AND STATEMENT OF THE THEOREM

Throughout this article, $\mathbf{K}$ will denote an imaginary quadratic field with discriminant $d_{\mathbf{K}}$, $h_{\mathbf{K}}$ the class number of $\mathcal{O}_{\mathbf{K}}$ and $|\mu_{\mathbf{K}}|$ the number of roots of unity in $\mathcal{O}_{\mathbf{K}}$. We will denote by $\zeta_{\mathbf{K}}$ the Dedekind zeta function of $\mathbf{K}$ and its residue at $s = 1$ by $\alpha_{\mathbf{K}}$. Further we use $\mathcal{P}_{\mathbf{K}}$ to denote the set of prime ideals of $\mathcal{O}_{\mathbf{K}}$ and $\mathcal{Q}$ to denote the set of all prime elements of $\mathcal{O}_{\mathbf{K}}$. We will denote by $\omega_{\mathbf{K}}(\mathfrak{b})$ the number of distinct prime ideals of $\mathcal{O}_{\mathbf{K}}$ which appear in the factorization of the ideal $\mathfrak{b}$ in $\mathcal{O}_{\mathbf{K}}$ and by $\pi_{\mathbf{K}}(x)$ the number of prime ideals of $\mathcal{O}_{\mathbf{K}}$ with norm at most x. The aim of this article is to prove a fully explicit generalisation of the Brun-Titchmarsh theorem for several linear forms taking values in $\mathcal{Q}$. This is a natural generalisation of the problem of finding an upper bound for the number of prime values that can be taken by a set of n linear forms simultaneously. This question has been addressed in considerable detail in literature.

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The study of such generalisations finds its origin in the twin prime and prime k-tuple conjectures. However the problem has been placed in the more general context of linear forms by Dickson's conjecture [5] which states the following.

Conjecture 1 (Dickson's conjecture [5]). *Given a set of n distinct irreducible linear polynomials $F_1, \ldots, F_n \in \mathbb{Z}[x]$ with positive leading coefficient, suppose that the product $\prod_{i=1}^{n} F_i(x)$ has no fixed prime divisor. Then the polynomials $F_i(x)$ simultaneously take prime values infinitely often.*

A quantitative version of Dickson's conjecture was given by Batemann and Horn [1, 2] in 1962. The precise form of the Bateman-Horn conjecture is as follows.

Conjecture 2 (Bateman-Horn conjecture [1, 2]). *Given a set of n distinct irreducible linear polynomials $F_1, \ldots F_n \in \mathbb{Z}[x]$ with positive leading coefficient, and suppose that the product $\prod_{i=1}^{n} F_i(x)$ has no fixed prime divisor. Then*

$$\sum_{\substack{1 \leqslant k \leqslant x \\ F_i(k) \text{ is prime } \forall i}} 1 = \prod_p \left\{ \left(1 - \frac{1}{p}\right)^{-n} \left(1 - \frac{\rho(p)}{p}\right) \right\} \cdot \int_2^x \frac{dt}{\log^n t} (1 + o(1))$$

as $x \to \infty$. Here $\rho(p)$ is the number of solutions of $\prod_{i=1}^{n} F_i(x) \equiv 0 \bmod p$.

The only case in which these conjectures have been resolved is in the case of a single linear polynomial which is nothing but the prime number theorem for primes in arithmetic progressions. For every other case finding even a lower bound in place of the asymptotic is notoriously difficult. However upper bounds close to the one suggested by the asymptotic are known using Selberg sieve techniques. For instance, one may find the following theorem in [9] (pages 157-159).

Theorem 1. *Given distinct irreducible linear polynomials $F_1, \ldots, F_n \in \mathbb{Z}[x]$ with positive leading coefficients, let $F(x) = \prod_{i=1}^{n} F_i(x)$. Further let $\rho(p)$ be the number of solutions modulo p of $F(x)$. If $\rho(p) < p$ for all primes p,*

$$\sum_{\substack{1 \leqslant k \leqslant x \\ F_i(k) \text{ is prime } \forall i}} 1 \leqslant \prod_p \left\{ \left(1 - \frac{1}{p}\right)^{-n} \left(1 - \frac{\rho(p)}{p}\right) \right\} \frac{2^n n! x}{\log^n x} \left(1 + \mathrm{O}_F\left(\frac{\log\log 3x}{\log x}\right)\right).$$

In this article, we show that an analogous bound can be obtained if we consider prime elements in an imaginary quadratic field instead of the rationals. Further our bounds are fully explicit. We present an application of such a bound in [10]. On the other hand this paper itself demonstrates an application of the main theorems of [8].

Theorem 2. *Let u be a positive real number, $n > 1$ be an integer and $a_i \in \mathcal{O}_{\mathbf{K}} \backslash \{0\}$'s for $1 \leqslant i \leqslant n$ be distinct. Assume that $(a_i \mathcal{O}_{\mathbf{K}},\ b_i \mathcal{O}_{\mathbf{K}}) = \mathcal{O}_{\mathbf{K}}$ for $1 \leqslant i \leqslant n$, $(a_i \mathcal{O}_{\mathbf{K}} : 1 \leqslant i \leqslant n) = \mathcal{O}_{\mathbf{K}}$ and*

$$E = \prod_{i=1}^{n} a_i \prod_{1 \leqslant i < j \leqslant n} (a_i b_j - a_j b_i) \neq 0.$$

Further assume that for any prime ideal $\mathfrak{p}$ of $\mathcal{O}_{\mathbf{K}}$, $\rho(\mathfrak{p})$ is the number of solutions of

$$\prod_{i=1}^{n}(a_i x + b_i) \equiv 0 \bmod \mathfrak{p}$$

and $\mathcal{Q}$ denotes the set of prime elements of $\mathcal{O}_{\mathbf{K}}$. Then for $u \geqslant [U(\mathbf{K}, a_1 b_1)]^4$, we have

$$\sum_{\substack{\mathfrak{N}(\alpha) \leqslant u \\ \forall i, b_i + a_i\alpha \in \mathcal{Q}}} 1 \leqslant \frac{5n!\,|\mu_{\mathbf{K}}|}{2\alpha_{\mathbf{K}}^{n-1} h_{\mathbf{K}}} \cdot \mathcal{S} \cdot \frac{u}{(\log C u^{\frac{1}{4}})^n},$$

where

$$U(\mathbf{K}, a_1 b_1) = \frac{\exp(18(n+1)L)}{C}, \qquad C = \frac{n!\pi}{3^{23n^3} n^{17n} \mathfrak{N}(a_1 b_1) \alpha_{\mathbf{K}}^{n} \sqrt{|d_{\mathbf{K}}|}},$$

$$L = 4n\omega_{\mathbf{K}}((E)) + 4n\omega_{\mathbf{K}}\Big(\prod_{\mathfrak{N}\mathfrak{p} \leqslant n} \mathfrak{p}\Big) + 20n^3 + n\frac{e^{76}|d_{\mathbf{K}}|^{1/3}(\log|d_{\mathbf{K}}|)^2}{\alpha_{\mathbf{K}}},$$

and

$$\mathcal{S} = \left(\prod_{\mathfrak{N}\mathfrak{p} \leqslant n} \frac{\mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p} - 1}\right)^n \prod_{\mathfrak{N}\mathfrak{p} > n} \left(1 - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right)\left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-n}.$$

The paper is organized as follows. In section 2, we will state some notations and preliminaries required for the proof of our main theorem. In the same section, we will also recall the results used from [8]. In section 3, we will prove some auxiliary lemmas and finally we will use them in section 4 to prove our theorem.

2. Notation and Preliminaries

Let $\mathbf{K}$ be an imaginary quadratic field and $\mathcal{O}_{\mathbf{K}}$ be its ring of integers. For an ideal $\mathfrak{q} \in \mathcal{O}_{\mathbf{K}}$, let $H_{\mathfrak{q}}(\mathbf{K})$ denote the ray class group modulo $\mathfrak{q}$ and $h_{\mathbf{K},\mathfrak{q}}$ denote its cardinality. When $\mathfrak{q} = \mathcal{O}_{\mathbf{K}}$, the ray class group modulo $\mathcal{O}_{\mathbf{K}}$ is $Cl_{\mathbf{K}}$. In this case, we denote $h_{\mathbf{K},\mathcal{O}_{\mathbf{K}}}$ by $h_{\mathbf{K}}$. Throughout the article, $\mathfrak{N}$ will denote the (absolute) norm, $\mathfrak{p}$ will denote a prime ideal in $\mathcal{O}_{\mathbf{K}}$ and p will denote a rational prime number. Further we use $\varphi(\mathfrak{q})$ to denote the Euler-phi function as defined below

$$\varphi(\mathfrak{q}) = \mathfrak{N}(\mathfrak{q}) \prod_{\mathfrak{p} | \mathfrak{q}} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right). \tag{1}$$

For any embedding σ of $\mathbf{K}$, the Minkowski embedding ψ of $\mathbf{K}$ to $\mathbb{R}^2$ maps x to $\Re(\sigma(x)), \Im(\sigma(x)))$. Let us begin with a counting theorem proved in [8].

Theorem 3. (Gun, Ramaré and Sivaraman) *Let $\mathfrak{a}, \mathfrak{q}$ be co-prime ideals of $\mathcal{O}_{\mathbf{K}}$, $\mathfrak{C}$ be the ideal class of $\mathfrak{a}\mathfrak{q}$ in the class group of $\mathcal{O}_{\mathbf{K}}$ and $\Lambda(\mathfrak{a}\mathfrak{q})$ be the lattice $\psi(\mathfrak{a}\mathfrak{q})$ in $\mathbb{R}^2$, where ψ is as defined above. Also let*

$$S_\beta\left(\mathfrak{a}, \mathfrak{q}, t^2\right) = \{\alpha \in \mathfrak{a} \,:\, |\psi(\alpha)|^2 \leqslant t^2,\ \alpha \equiv \beta \bmod \mathfrak{q}\}$$

for some fix $\beta \in \mathcal{O}_{\mathbf{K}}$. Then for any real number $t \geqslant 1$, we have

$$|S_\beta(\mathfrak{a}, \mathfrak{q}, t^2)| = \frac{(2\pi)}{\sqrt{|d_{\mathbf{K}}|}\,\mathfrak{N}(\mathfrak{a}\mathfrak{q})} t^2 + \mathrm{O}^*\left(\frac{10^{13.66}\mathfrak{N}(\mathfrak{C}^{-1})}{|\mathfrak{N}(\mathfrak{a}\mathfrak{q})|^{\frac{1}{2}}} t + 1\right), \tag{2}$$

where

$$\mathfrak{N}(\mathfrak{C}^{-1}) = max_{\mathfrak{b}\in\mathfrak{C}^{-1}} \frac{1}{|\mathfrak{N}(\mathfrak{b})|^{\frac{1}{2}}}.$$

One can ignore 1 in the error term when $\mathfrak{q} = \mathcal{O}_{\mathbf{K}}$.

For an arithmetic function f and a positive arithmetic function g, $f(z) = \mathcal{O}^*(g(z))$ implies that $|f(z)| \leqslant g(z)$.

The Dedekind zeta-function. For $\Re s = \sigma > 1$, the Dedekind zeta-function is defined by

$$\zeta_{\mathbf{K}}(s) = \sum_{\mathfrak{a}\subseteq\mathcal{O}_{\mathbf{K}}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s},$$

where $\mathfrak{a}$ ranges over the integral ideals of $\mathcal{O}_{\mathbf{K}}$. It has only a simple pole at $s = 1$ of residue $\alpha_{\mathbf{K}}$, say. When $\mathbf{K}$ is an imaginary quadratic field, we know from the analytic class number formula that

$$\alpha_{\mathbf{K}} = \frac{2\pi h_{\mathbf{K}}}{|\mu_{\mathbf{K}}|\sqrt{|d_{\mathbf{K}}|}}, \tag{3}$$

where $h_{\mathbf{K}}, d_{\mathbf{K}}$ and $|\mu_{\mathbf{K}}|$ are as before. We now quote a result from [4] which will be used to prove our theorem.

Lemma 4. (Deshouillers, Gun, Ramaré and Sivaraman) *If $\alpha_{\mathbf{K}}$ is the residue at $s = 1$ of the Dedekind zeta function of $\mathbf{K}$, then we have*

$$\frac{36}{100\sqrt{|d_{\mathbf{K}}|}} \leqslant \alpha_{\mathbf{K}} \leqslant 6(2\pi^2/5)^2|d_{\mathbf{K}}|^{1/4}.$$

The next lemma is a result from [3] and is used to estimate the error term in Theorem 3.

Lemma 5. (Debaene) *Let $\mathfrak{b}_1, \mathfrak{b}_2, \cdots$ be integral ideals of $\mathcal{O}_{\mathbf{K}}$, ordered such that $\mathfrak{N}(\mathfrak{b}_1) \leqslant \mathfrak{N}(\mathfrak{b}_2)\cdots$. Then for any real number $y \geqslant 2$*

$$\sum_{i=1}^{y} \mathfrak{N}(\mathfrak{b}_i)^{-\frac{1}{2}} \leqslant 12 y^{\frac{1}{2}} (\log y)^{\frac{1}{2}}.$$

Finally we recall two estimates which will be used in due course of our proof.

Lemma 6. (Debaene [3]) *For any real number $y \geqslant 16$, we have*

$$\sum_{p\leqslant y} \frac{1}{p} \leqslant 0.666 + \log\log y.$$

Lemma 7. (Rosser and Schoenfeld [12]) *For any real number $y \geqslant 1$, we have*

$$\sum_{p \leqslant y} \frac{1}{p} \geqslant \log\log y.$$

2.1. **Counting the total number of ideals.** Applying Lemma 5, we derive the following corollary from Theorem 3.

Corollary 8. *Let $\mathbf{K}$ be an imaginary quadratic field. For any real number $x \geqslant 1$, we have*

$$\sum_{\substack{\mathfrak{a} \subset \mathcal{O}_{\mathbf{K}},\\ \mathfrak{N}\mathfrak{a} \leqslant x}} 1 = \alpha_{\mathbf{K}} x + \mathrm{O}^*\left(10^{15}(h_{\mathbf{K}} \log(3h_{\mathbf{K}}))^{\frac{1}{2}} x^{\frac{1}{2}}\right).$$

Proof. For any class $[\mathfrak{C}]$ in the class group $Cl_{\mathbf{K}}$ of $\mathcal{O}_{\mathbf{K}}$, choose an integral ideal $\mathfrak{b}_{\mathfrak{C}} \in [\mathfrak{C}^{-1}]$. From Theorem 3, we have

$$\begin{aligned}\sum_{\substack{\mathfrak{a} \subset \mathcal{O}_{\mathbf{K}},\\ \mathfrak{N}\mathfrak{a} \leqslant x}} 1 = \sum_{\mathfrak{C} \in Cl_{\mathbf{K}}} \sum_{\substack{\mathfrak{a} \in [\mathfrak{C}] \cap \mathcal{O}_{\mathbf{K}},\\ \mathfrak{N}\mathfrak{a} \leqslant x}} 1 &= \sum_{\mathfrak{C} \in Cl_{\mathbf{K}}} \frac{1}{|\mu_{\mathbf{K}}|} \left|\left\{\alpha \in \mathfrak{b}_{\mathfrak{C}} : |\phi(\alpha)|^2 \leqslant x \mathfrak{N}\mathfrak{b}_{\mathfrak{C}}\right\}\right| \\ &= \frac{2\pi h_{\mathbf{K}} x}{|\mu_{\mathbf{K}}|\sqrt{|d_{\mathbf{K}}|}} + \mathrm{O}^*\left(10^{13.66}\sqrt{x} \sum_{\mathfrak{C} \in Cl_{\mathbf{K}}} \mathfrak{N}(\mathfrak{C}^{-1})\right).\end{aligned}$$

To majorize $\sum_{\mathfrak{C} \in Cl_{\mathbf{K}}} \mathfrak{N}(\mathfrak{C}^{-1})$, we apply Lemma 5 with $y = 3h_{\mathbf{K}}$. This completes the proof of Corollary 8. □

3. Some intermediate lemmas

3.1. **Selberg sieve.** Let n be an integer greater than or equal to 2, $a_i x + b_i$ for $1 \leqslant i \leqslant n$ be n distinct linear forms with $a_i, b_i \in \mathcal{O}_{\mathbf{K}} \backslash \{0\}$, $(a_i \mathcal{O}_{\mathbf{K}},\ b_i \mathcal{O}_{\mathbf{K}}) = \mathcal{O}_{\mathbf{K}}$ and $(a_i \mathcal{O}_{\mathbf{K}} : 1 \leqslant i \leqslant n) = \mathcal{O}_{\mathbf{K}}$. We further assume that

$$E = \prod_{i=1}^{n} a_i \prod_{1 \leqslant i < j \leqslant n} (a_i b_j - a_j b_i) \quad \text{and} \quad H = \prod_{\mathfrak{N}\mathfrak{p} \leqslant n} \mathfrak{p}.$$

For an integral ideal $\mathfrak{b}$, let $\rho(\mathfrak{b})$ denote the number of solutions of

$$F(x) = \prod_{i=1}^{n} (a_i x + b_i) \equiv 0 \bmod \mathfrak{b}.$$

Applying Chinese remainder theorem, it follows that $\rho(\mathfrak{b})$ is a multiplicative function. Further, we observe that for any prime $\mathfrak{p}$, $\rho(\mathfrak{p}) < \mathfrak{N}\mathfrak{p}$ when $\mathfrak{N}\mathfrak{p} > n$. Let us define the multiplicative functions

$$f(\mathfrak{b}) = \frac{\mathfrak{N}\mathfrak{b}}{\rho(\mathfrak{b})} \quad \text{and} \quad f_1(\mathfrak{b}) = \sum_{\substack{\mathfrak{a} \mid \mathfrak{b}\\ \mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}}}} \mu(\mathfrak{a}) f\left(\frac{\mathfrak{b}}{\mathfrak{a}}\right). \tag{4}$$

We may assume that $\rho(\mathfrak{p}) < \mathfrak{N}\mathfrak{p}$ when $\mathfrak{N}\mathfrak{p} \leqslant z$ since otherwise no prime of norm greater than z is to be counted in our sum. Hence $f_1 > 0$ on the set of non zero square free integral ideals co-prime to H. Also $f(\mathcal{O}_{\mathbf{K}}) = 1$. Further, for an ideal $\mathfrak{e}$ of $\mathcal{O}_{\mathbf{K}}$ co-prime to H, we define

$$\mathcal{P}_{\mathbf{K}}(z) = \prod_{n < \mathfrak{N}\mathfrak{p} \leqslant z} \mathfrak{p}, \; S_{\mathfrak{e}}(z) = \sum_{\substack{\mathfrak{N}(\mathfrak{a}) \leqslant z, \\ (\mathfrak{a},\; \mathfrak{e}H) = \mathcal{O}_K}} \frac{\mu^2(\mathfrak{a})}{f_1(\mathfrak{a})}, \; G(z) = S_{\mathcal{O}_{\mathbf{K}}}(z) \text{ and } \lambda_{\mathfrak{e}} = \mu(\mathfrak{e}) \frac{f(\mathfrak{e}) S_{\mathfrak{e}}(\frac{z}{\mathfrak{N}(\mathfrak{e})})}{f_1(\mathfrak{e}) G(z)}.$$

Proposition 9. *For any ideal $\mathfrak{b} \mid \mathcal{P}_{\mathbf{K}}(z)$, we have $|\lambda_{\mathfrak{b}}| \leqslant 1$.*

Proof. For an integral ideal $\mathfrak{b}$ dividing $\mathcal{P}_{\mathbf{K}}(z)$, we have

$$\begin{aligned} S_{\mathcal{O}_{\mathbf{K}}}(z) &= \sum_{\mathfrak{c} \mid \mathfrak{b}} \frac{\mu^2(\mathfrak{c})}{f_1(\mathfrak{c})} \sum_{\substack{\mathfrak{N}(\mathfrak{a}) \leqslant \frac{z}{\mathfrak{N}(\mathfrak{c})} \\ (\mathfrak{a}, \mathfrak{b}H) = \mathcal{O}_{\mathbf{K}} \\ \mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{a})}{f_1(\mathfrak{a})} \geqslant \sum_{\mathfrak{c} \mid \mathfrak{b}} \frac{\mu^2(\mathfrak{c})}{f_1(\mathfrak{c})} \sum_{\substack{\mathfrak{N}(\mathfrak{a}) \leqslant \frac{z}{\mathfrak{N}(\mathfrak{b})} \\ (\mathfrak{a}, \mathfrak{b}H) = \mathcal{O}_{\mathbf{K}} \\ \mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{a})}{f_1(\mathfrak{a})} \\ &= S_{\mathfrak{b}}\left(\frac{z}{\mathfrak{N}(\mathfrak{b})}\right) \sum_{\mathfrak{c} \mid \mathfrak{b}} \frac{\mu^2(\mathfrak{c})}{f_1(\mathfrak{c})} = \frac{f(\mathfrak{b})}{f_1(\mathfrak{b})} S_{\mathfrak{b}}\left(\frac{z}{\mathfrak{N}(\mathfrak{b})}\right). \end{aligned}$$

The last step follows from the fact that $\mathfrak{b}$ is square-free and co-prime to H. To see this, note that

$$\sum_{\mathfrak{a} \mid \mathfrak{b}} \frac{\mu^2(\mathfrak{a})}{f_1(\mathfrak{a})} = \prod_{\mathfrak{p} \mid \mathfrak{b}} \left(1 + \frac{1}{f_1(\mathfrak{p})}\right) = \frac{\sum_{\mathfrak{a} \mid \mathfrak{b}} f_1(\mathfrak{a})}{f_1(\mathfrak{b})} = \frac{f(\mathfrak{b})}{f_1(\mathfrak{b})}.$$

This completes the proof of the lemma. □

We now recall a special case of a result of Garcia and Lee [7] .

Theorem 10. *Let $\mathbf{K}$ be an imaginary quadratic field and $x \geqslant 2$. We have*

$$\sum_{\mathfrak{N}\mathfrak{p} \leqslant x} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} = \log x + \mathrm{O}^*\left(3 + \frac{e^{75} |d_{\mathbf{K}}|^{1/3} (\log |d_{\mathbf{K}}|)^2}{\alpha_{\mathbf{K}}}\right).$$

Using the above theorem, we can now prove the following asymptotic.

Lemma 11. *Let $x \geqslant 2$ be a real number. The sum*

$$\sum_{n < \mathfrak{N}\mathfrak{p} \leqslant x} \frac{\rho(\mathfrak{p}) \log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} = n \log x + \mathrm{O}^*\left(n\left(\omega_{\mathbf{K}}(E) + \omega_{\mathbf{K}}(H) + 3 + \frac{e^{75} |d_{\mathbf{K}}|^{1/3} (\log |d_{\mathbf{K}}|)^2}{\alpha_{\mathbf{K}}}\right)\right).$$

Proof. It follows from the definition of ρ, H and E that

$$\sum_{n < \mathfrak{N}\mathfrak{p} \leqslant x} \frac{\rho(\mathfrak{p}) \log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} = \sum_{\mathfrak{N}\mathfrak{p} \leqslant x} \frac{n \log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} + \mathrm{O}^*(n(\omega_{\mathbf{K}}(E) + \omega_{\mathbf{K}}(H))),$$

where $\omega_{\mathbf{K}}(E)$ denotes the number of distinct prime ideals of $\mathbf{K}$ dividing the ideal (E) in $\mathbf{K}$. Thus we have the lemma.

□

3.1.1. *An estimate to control the error term.*

Lemma 12. *We have*

$$\sum_{\substack{\mathfrak{b}_1,\mathfrak{b}_2|\mathcal{P}_{\mathbf{K}}(z),\\ \mathfrak{N}\mathfrak{b}_i\leqslant z}} |\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}|\frac{\rho([\mathfrak{b}_1,\mathfrak{b}_2])}{\sqrt{\mathfrak{N}([\mathfrak{b}_1,\mathfrak{b}_2])}} \leqslant (3n)^{4\pi_{\mathbf{K}}(2n)}\zeta_{\mathbf{K}}\left(\frac{3}{2}\right)^{8n} z.$$

Proof. We consider the sum

$$\sum_{\substack{\mathfrak{b}_1,\mathfrak{b}_2|\mathcal{P}_{\mathbf{K}}(z),\\ \mathfrak{N}\mathfrak{b}_i\leqslant z}} |\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}|\frac{\rho([\mathfrak{b}_1,\mathfrak{b}_2])}{\sqrt{\mathfrak{N}([\mathfrak{b}_1,\mathfrak{b}_2])}} = \sum_{\substack{\partial|\mathcal{P}_{\mathbf{K}}(z),\\ \mathfrak{N}\partial\leqslant z}} \frac{\sqrt{\mathfrak{N}\partial}}{\rho(\partial)} \sum_{\substack{\mathfrak{b}_i|\mathcal{P}_{\mathbf{K}}(z),\\ \partial=(\mathfrak{b}_1,\mathfrak{b}_2)\\ \mathfrak{N}\mathfrak{b}_i\leqslant z}} \frac{|\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}|\rho(\mathfrak{b}_1)\rho(\mathfrak{b}_2)}{\sqrt{\mathfrak{N}(\mathfrak{b}_1\mathfrak{b}_2)}}.$$

From the expression of $\lambda_{\mathfrak{b}}$ and with $y = z/\mathfrak{N}\partial$, we get

$$\begin{aligned}
\sum_{\substack{\mathfrak{N}\mathfrak{c}\leqslant y,\\ (\mathfrak{c},\partial H)=1}} \frac{|\lambda_{\partial\mathfrak{c}}|\rho(\mathfrak{c})}{\sqrt{\mathfrak{N}\mathfrak{c}}} &= G(z)^{-1} \sum_{\substack{\mathfrak{N}\mathfrak{c}\leqslant y,\\ (\mathfrak{c},\partial H)=\mathcal{O}_{\mathbf{K}}}} \mu^2(\mathfrak{c})\frac{\sqrt{\mathfrak{N}\mathfrak{c}}\mathfrak{N}\partial}{\rho(\partial)f_1(\mathfrak{c}\partial)} \sum_{\substack{\mathfrak{N}\mathfrak{m}\leqslant y/\mathfrak{N}(\mathfrak{c}),\\ (\mathfrak{m},\mathfrak{c}\partial H)=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{m})}{f_1(\mathfrak{m})}\\
&\leqslant \frac{\mathfrak{N}\partial}{G(z)\rho(\partial)f_1(\partial)} \sum_{\substack{\mathfrak{N}\mathfrak{m}\leqslant y\\ (\mathfrak{m},H)=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{m})}{f_1(\mathfrak{m})} \sum_{\substack{\mathfrak{N}\mathfrak{c}\leqslant y/\mathfrak{N}\mathfrak{m}\\ (\mathfrak{c},H)=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{c})\sqrt{\mathfrak{N}\mathfrak{c}}}{f_1(\mathfrak{c})}.\\
&\leqslant \frac{\sqrt{y}\,\mathfrak{N}\partial}{G(z)\rho(\partial)f_1(\partial)} \sum_{\substack{\mathfrak{N}\mathfrak{m}\leqslant y\\ (\mathfrak{m},H)=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{m})}{f_1(\mathfrak{m})\sqrt{\mathfrak{N}\mathfrak{m}}}G(y) \leqslant \frac{\sqrt{y}\,\mathfrak{N}\partial}{\rho(\partial)f_1(\partial)} \prod_{\mathfrak{N}\mathfrak{p}>n}\left(1+\frac{\rho(\mathfrak{p})}{\sqrt{\mathfrak{N}\mathfrak{p}}(\mathfrak{N}\mathfrak{p}-\rho(\mathfrak{p}))}\right).
\end{aligned}$$

We thus get

$$\begin{aligned}
\sum_{\substack{\mathfrak{b}_1,\mathfrak{b}_2|\mathcal{P}_{\mathbf{K}}(z),\\ \mathfrak{N}\mathfrak{b}_i\leqslant z}} |\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}|\frac{\rho([\mathfrak{b}_1,\mathfrak{b}_2])}{\sqrt{\mathfrak{N}([\mathfrak{b}_1,\mathfrak{b}_2])}} &\leqslant \sum_{\substack{\partial|\mathcal{P}_{\mathbf{K}}(z)\\ \mathfrak{N}\partial\leqslant z}} \frac{\rho(\partial)}{\sqrt{\mathfrak{N}\partial}}\left(\frac{\sqrt{z\,\mathfrak{N}\partial}}{\rho(\partial)f_1(\partial)}\prod_{\mathfrak{N}\mathfrak{p}>n}\left(1+\frac{\rho(\mathfrak{p})}{\sqrt{\mathfrak{N}\mathfrak{p}}(\mathfrak{N}\mathfrak{p}-\rho(\mathfrak{p}))}\right)\right)^2\\
&\leqslant z\prod_{\mathfrak{N}\mathfrak{p}>n}\left(1+\frac{\rho(\mathfrak{p})}{\sqrt{\mathfrak{N}\mathfrak{p}}(\mathfrak{N}\mathfrak{p}-\rho(\mathfrak{p}))}\right)^2\left(1+\frac{\rho(\mathfrak{p})\sqrt{\mathfrak{N}\mathfrak{p}}}{(\mathfrak{N}\mathfrak{p}-\rho(\mathfrak{p}))^2}\right)
\end{aligned}$$

Note that

$$\prod_{\mathfrak{N}\mathfrak{p}>n}\left(1+\frac{\rho(\mathfrak{p})}{\sqrt{\mathfrak{N}\mathfrak{p}}(\mathfrak{N}\mathfrak{p}-\rho(\mathfrak{p}))}\right) \leqslant (2n)^{\pi_{\mathbf{K}}(2n)}\prod_{\substack{\mathfrak{p}\\ \mathfrak{N}\mathfrak{p}>2n}}\left(1+\frac{2n}{\mathfrak{N}\mathfrak{p}^{\frac{3}{2}}}\right) \leqslant (2n)^{\pi_{\mathbf{K}}(2n)}\zeta_{\mathbf{K}}\left(\frac{3}{2}\right)^{2n}.$$

Similarly

$$\prod_{\mathfrak{N}\mathfrak{p}>n}\left(1+\frac{\rho(\mathfrak{p})\sqrt{\mathfrak{N}\mathfrak{p}}}{(\mathfrak{N}\mathfrak{p}-\rho(\mathfrak{p}))^2}\right) \leqslant \prod_{n<\mathfrak{N}\mathfrak{p}\leqslant 2n}(1+n\sqrt{\mathfrak{N}\mathfrak{p}})\prod_{\mathfrak{N}\mathfrak{p}>2n}\left(1+\frac{4n}{\mathfrak{N}\mathfrak{p}^{\frac{3}{2}}}\right) \leqslant (3n)^{\frac{3\pi_{\mathbf{K}}(2n)}{2}}\zeta_{\mathbf{K}}\left(\frac{3}{2}\right)^{4n}.$$

This completes the proof of the lemma. □

3.1.2. *Estimating G(z).* We redo the proof of Fainleib-Levin [6] as described in Halberstam-Richert [9] in the number field setting with the additional condition $\partial | \mathcal{P}_{\mathbf{K}}(z)$. It can also be done using the methods of Theorem 13.3 of [11]. Let

$$G(x,z) = \sum_{\substack{\partial | \mathcal{P}_{\mathbf{K}}(z) \\ \mathfrak{N}\partial \leqslant x}} \frac{\mu^2(\partial)}{f_1(\partial)} \quad \text{and} \quad G_{\mathfrak{p}}(x,z) = \sum_{\substack{\partial | \mathcal{P}_{\mathbf{K}}(z) \\ \mathfrak{N}\partial \leqslant x \\ (\partial, \mathfrak{p}) = \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\partial)}{f_1(\partial)}.$$

Lemma 13. *We have*

$$\left(1 - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right) G_{\mathfrak{p}}\left(\frac{x}{\mathfrak{N}\mathfrak{p}}, z\right) = \left(1 - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right) G\left(\frac{x}{\mathfrak{N}\mathfrak{p}}, z\right) - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}} G_{\mathfrak{p}}\left(\frac{x}{\mathfrak{N}\mathfrak{p}^2}, z\right).$$

Proof. For an integral ideal ∂ co-prime to H, let $h(\partial) = \frac{\mu^2(\partial)}{f_1(\partial)}$. The function h is multiplicative. From the definition of $G(x,z)$, we have

$$G(x,z) = \sum_{\substack{\partial | \mathcal{P}_{\mathbf{K}}(z) \\ \mathfrak{N}\partial \leqslant x}} h(\partial) = \sum_{\substack{\partial | \mathcal{P}_{\mathbf{K}}(z) \\ \mathfrak{N}\partial \leqslant x \\ (\partial, \mathfrak{p}) = \mathcal{O}_{\mathbf{K}}}} h(\partial) \; + \; h(\mathfrak{p}) \sum_{\substack{\partial | \mathcal{P}_{\mathbf{K}}(z) \\ \mathfrak{N}\partial \leqslant \frac{x}{\mathfrak{N}\mathfrak{p}} \\ (\partial, \mathfrak{p}) = \mathcal{O}_{\mathbf{K}}}} h(\partial).$$

Multiplying both sides with $(1 - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}})$, we get

$$\left(1 - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right) G(x,z) = \left(1 - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right) G_{\mathfrak{p}}(x,z) + \left(1 - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right) h(\mathfrak{p}) G_{\mathfrak{p}}\left(\frac{x}{\mathfrak{N}\mathfrak{p}}, z\right).$$

4 However we note that

$$\left(1 - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right) h(\mathfrak{p}) = \left(1 - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right) \frac{1}{f_1(\mathfrak{p})} = \left(1 - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right) \frac{1}{f(\mathfrak{p}) - 1} = \frac{1}{f(\mathfrak{p})}. \tag{5}$$

This now gives us

$$\left(1 - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right) G(x,z) = \left(1 - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right) G_{\mathfrak{p}}(x,z) + \frac{1}{f(\mathfrak{p})} G_{\mathfrak{p}}\left(\frac{x}{\mathfrak{N}\mathfrak{p}}, z\right).$$

Replacing x by $\frac{x}{\mathfrak{N}\mathfrak{p}}$ in $G(x,z)$, we get

$$\left(1 - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right) G\left(\frac{x}{\mathfrak{N}\mathfrak{p}}, z\right) = \left(1 - \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right) G_{\mathfrak{p}}\left(\frac{x}{\mathfrak{N}\mathfrak{p}}, z\right) + \frac{1}{f(\mathfrak{p})} G_{\mathfrak{p}}\left(\frac{x}{\mathfrak{N}\mathfrak{p}^2}, z\right).$$

1 Thus we have the lemma. □

Lemma 14. *For an integral ideal ∂ and real number $x > \mathfrak{N}\partial$ we have*

$$\sum_{\substack{\sqrt{\frac{x}{\mathfrak{N}\partial}} < \mathfrak{N}\mathfrak{p} \leqslant \frac{x}{\mathfrak{N}\partial}, \\ \mathfrak{p} \nmid \partial}} h(\mathfrak{p}) \;\leqslant\; n(\pi_{\mathbf{K}}(2n) + 9),$$

2 *where $\pi_{\mathbf{K}}(x)$ denotes the number of prime ideals of $\mathcal{O}_{\mathbf{K}}$ with norm at most x.*

Proof. We have

$$\sum_{\substack{\sqrt{\frac{x}{\mathfrak{N}\partial}}<\mathfrak{N}\mathfrak{p}\leqslant\frac{x}{\mathfrak{N}\partial},\\ \mathfrak{p}\nmid\partial H}} h(\mathfrak{p}) \leqslant \sum_{\substack{\sqrt{\frac{x}{\mathfrak{N}\partial}}<\mathfrak{N}\mathfrak{p}\leqslant\frac{x}{\mathfrak{N}\partial},\\ \mathfrak{p}\nmid\partial H}} \frac{n}{\mathfrak{N}\mathfrak{p}-\rho(\mathfrak{p})} \leqslant \sum_{\substack{\mathfrak{N}\mathfrak{p}<2n\\ \mathfrak{p}\nmid H}} \frac{n}{\mathfrak{N}\mathfrak{p}-\rho(\mathfrak{p})} + \sum_{\substack{\sqrt{\frac{x}{\mathfrak{N}\partial}}<\mathfrak{N}\mathfrak{p}\leqslant\frac{x}{\mathfrak{N}\partial},\\ \mathfrak{p}\nmid\partial}} \frac{2n}{\mathfrak{N}\mathfrak{p}}.$$

This gives us

$$\sum_{\substack{\sqrt{\frac{x}{\mathfrak{N}\partial}}<\mathfrak{N}\mathfrak{p}\leqslant\frac{x}{\mathfrak{N}\partial},\\ \mathfrak{p}\nmid\partial H}} h(\mathfrak{p}) \leqslant n\pi_{\mathbf{K}}(2n) + 2\sum_{\sqrt{\frac{x}{\mathfrak{N}\partial}}<p\leqslant\frac{x}{\mathfrak{N}\partial}} \frac{2n}{p} + \sum_{p\leqslant\sqrt{\frac{x}{\mathfrak{N}\partial}}} \frac{2n}{p^2}.$$

Note that

$$\sum_{\sqrt{\frac{x}{\mathfrak{N}\partial}}<p\leqslant\frac{x}{\mathfrak{N}\partial}} \frac{2n}{p} = \sum_{p\leqslant\frac{x}{\mathfrak{N}\partial}} \frac{2n}{p} - \sum_{p\leqslant\sqrt{\frac{x}{\mathfrak{N}\partial}}} \frac{2n}{p}.$$

The first sum is estimated using a result of Debaene [3] (see Lemma 6) and the second using a result of Rosser and Schoenfeld [12] (see Lemma 7). This gives us for $x \geqslant 16\mathfrak{N}\partial$,

$$\sum_{p\leqslant\frac{x}{\mathfrak{N}\partial}} \frac{2n}{p} - \sum_{p\leqslant\sqrt{\frac{x}{\mathfrak{N}\partial}}} \frac{2n}{p} \leqslant 2n\left(0.666+\log\log\frac{x}{\mathfrak{N}\partial}-\log\log\sqrt{\frac{x}{\mathfrak{N}\partial}}\right) \leqslant 2n\,(0.666+\log 2) \leqslant 2.8n.$$

If $x < 16\mathfrak{N}\partial$

$$\sum_{p\leqslant\frac{x}{\mathfrak{N}\partial}} \frac{2n}{p} - \sum_{p\leqslant\sqrt{\frac{x}{\mathfrak{N}\partial}}} \frac{2n}{p} \leqslant \sum_{p<16} \frac{2n}{p} \leqslant 2.7n.$$

3 □

Let

$$T(x,z) = \int_1^x G(t,z)\,\frac{dt}{t}.$$

It follows from the definition of $G(t,z)$ that

$$T(x,z) = \int_1^x \sum_{\substack{\mathfrak{N}\partial\leqslant t,\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\,\frac{dt}{t} = \sum_{\substack{\mathfrak{N}\partial\leqslant x,\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\int_{\mathfrak{N}\partial}^x \frac{dt}{t} = \sum_{\substack{\mathfrak{N}\partial\leqslant x,\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\log\frac{x}{\mathfrak{N}\partial}.$$

4 **Lemma 15.** *Let $z \geqslant 1$ be a real number. The sum*

$$\sum_{\substack{\mathfrak{N}\partial\leqslant x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\log\mathfrak{N}\partial = nT(x,z) - nT\left(\frac{x}{z},z\right)$$
$$+ \mathrm{O}^*\left(\left(\omega_{\mathbf{K}}(E)+\omega_{\mathbf{K}}(H)+10n^2+\frac{e^{75}|d_{\mathbf{K}}|^{1/3}(\log|d_{\mathbf{K}}|)^2}{\alpha_{\mathbf{K}}}\right)nG(x,z)\right).$$

Proof. We have

$$S = \sum_{\substack{\mathfrak{N}\partial\leqslant x,\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\sum_{\mathfrak{p}|\partial}\log\mathfrak{N}\mathfrak{p} = \sum_{n<\mathfrak{N}\mathfrak{p}\leqslant z} h(\mathfrak{p})\log\mathfrak{N}\mathfrak{p}\sum_{\substack{\mathfrak{N}\mathfrak{m}\leqslant\frac{x}{\mathfrak{N}\mathfrak{p}}\\ \mathfrak{m}|\mathcal{P}_{\mathbf{K}}(z),\\ (\mathfrak{m},\mathfrak{p})=\mathcal{O}_{\mathbf{K}}}} h(\mathfrak{m}) = \sum_{n<\mathfrak{N}\mathfrak{p}\leqslant z} h(\mathfrak{p})G_{\mathfrak{p}}\left(\frac{x}{\mathfrak{N}\mathfrak{p}},z\right)\log\mathfrak{N}\mathfrak{p}.$$

Applying Lemma 13 and using (5), we get

$$S=\sum_{n<\mathfrak{N}\mathfrak{p}\leqslant z}\frac{\rho(\mathfrak{p})\log\mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}}G\left(\frac{x}{\mathfrak{N}\mathfrak{p}},z\right)\;+\sum_{n<\mathfrak{N}\mathfrak{p}\leqslant z}\frac{\rho(\mathfrak{p})h(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\log\mathfrak{N}\mathfrak{p}\sum_{\substack{\frac{x}{\mathfrak{N}\mathfrak{p}^2}<\mathfrak{N}\mathfrak{m}\leqslant\frac{x}{\mathfrak{N}\mathfrak{p}}\\ \mathfrak{m}|\mathcal{P}_{\mathbf{K}}(z),\\ (\mathfrak{m},\mathfrak{p})=\mathcal{O}_{\mathbf{K}}}}h(\mathfrak{m}).$$

Using the definition of $G(x,z)$ in the first sum and interchanging the summations, we get

$$S=\sum_{\substack{\mathfrak{N}\partial\leqslant x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}}h(\partial)\sum_{n<\mathfrak{N}\mathfrak{p}\leqslant\min\left(\frac{x}{\mathfrak{N}\partial},z\right)}\frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\log\mathfrak{N}\mathfrak{p}\;+\sum_{\substack{\frac{x}{z^2}<\mathfrak{N}\partial\leqslant x,\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}}h(\partial)\sum_{\substack{\sqrt{\frac{x}{\mathfrak{N}\partial}}<\mathfrak{N}\mathfrak{p}\leqslant\min(\frac{x}{\mathfrak{N}\partial},z),\\ (\mathfrak{p},\partial H)=\mathcal{O}_{\mathbf{K}}}}\frac{\rho(\mathfrak{p})h(\mathfrak{p})\log\mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}}.$$

Applying Lemma 14, we get

$$\sum_{\substack{\sqrt{\frac{x}{\mathfrak{N}\partial}}<\mathfrak{N}\mathfrak{p}\leqslant\min(\frac{x}{\mathfrak{N}\partial},z)\\ (\mathfrak{p},\partial H)=\mathcal{O}_{\mathbf{K}}}}\frac{\rho(\mathfrak{p})h(\mathfrak{p})\log\mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}}\;\leqslant\;n\sum_{\substack{\sqrt{\frac{x}{\mathfrak{N}\partial}}<\mathfrak{N}\mathfrak{p}\leqslant\min(\frac{x}{\mathfrak{N}\partial},z)\\ \mathfrak{p}\nmid\partial H}}h(\mathfrak{p})\leqslant n^2(\pi_{\mathbf{K}}(2n)+9).$$

Combining the above we get

$$S=\sum_{\substack{\mathfrak{N}\partial\leqslant x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}}h(\partial)\sum_{n<\mathfrak{N}\mathfrak{p}\leqslant\min\left(\frac{x}{\mathfrak{N}\partial},z\right)}\frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\log\mathfrak{N}\mathfrak{p}\;+\;\mathrm{O}^*(n^2(\pi_{\mathbf{K}}(2n)+9)\,G(x,z)).$$

For the first term, we get

$$\sum_{\substack{\mathfrak{N}\partial\leqslant x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}}h(\partial)\sum_{n<\mathfrak{N}\mathfrak{p}\leqslant\min\left(\frac{x}{\mathfrak{N}\partial},z\right)}\frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\log\mathfrak{N}\mathfrak{p}\;=\sum_{\substack{\mathfrak{N}\partial\leqslant\frac{x}{z}\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}}h(\partial)\sum_{n<\mathfrak{N}\mathfrak{p}\leqslant z}\frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\log\mathfrak{N}\mathfrak{p}+\sum_{\substack{\frac{x}{z}<\mathfrak{N}\partial\leqslant x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}}h(\partial)\sum_{n<\mathfrak{N}\mathfrak{p}\leqslant\frac{x}{\mathfrak{N}\partial}}\frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\log\mathfrak{N}\mathfrak{p}.$$

We now apply Lemma 11 to deduce

$$\begin{aligned}\sum_{\substack{\mathfrak{N}\partial\leqslant x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}}h(\partial)\sum_{n<\mathfrak{N}\mathfrak{p}\leqslant\min\left(\frac{x}{\mathfrak{N}\partial},z\right)}\frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\log\mathfrak{N}\mathfrak{p}\;&=\;n\sum_{\substack{\mathfrak{N}\partial\leqslant\frac{x}{z},\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}}h(\partial)\log z\;+\;n\sum_{\substack{\frac{x}{z}<\mathfrak{N}\partial\leqslant x,\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}}h(\partial)\log\frac{x}{\mathfrak{N}\partial}\\ &+\;\mathrm{O}^*\left(n\left(\omega_{\mathbf{K}}(E)+\omega_{\mathbf{K}}(H)+3+\frac{e^{75}|d_{\mathbf{K}}|^{1/3}(\log|d_{\mathbf{K}}|)^2}{\alpha_{\mathbf{K}}}\right)G(x,z)\right).\end{aligned}$$

Combining the above, we get

$$\begin{aligned}S\;&=\;n\sum_{\substack{\mathfrak{N}\partial\leqslant x\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}}h(\partial)\log\frac{x}{\mathfrak{N}\partial}-n\sum_{\substack{\mathfrak{N}\partial\leqslant\frac{x}{z},\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}}h(\partial)\log\frac{x/z}{\mathfrak{N}\partial}\\ &+\;\mathrm{O}^*\left(n\left(\omega_{\mathbf{K}}(E)+\omega_{\mathbf{K}}(H)+3+n(\pi_{\mathbf{K}}(2n)+9)+\frac{e^{75}|d_{\mathbf{K}}|^{1/3}(\log|d_{\mathbf{K}}|)^2}{\alpha_{\mathbf{K}}}\right)G(x,z)\right)\\ &=\;nT(x,z)-nT(x/z,z)+\mathrm{O}^*\left(n\left(\omega_{\mathbf{K}}(E)+\omega_{\mathbf{K}}(H)+10n^2+\frac{e^{75}|d_{\mathbf{K}}|^{1/3}(\log|d_{\mathbf{K}}|)^2}{\alpha_{\mathbf{K}}}\right)G(x,z)\right).\end{aligned}$$

□

Note that $G(z,z)=G(z)$ and $T(z,z)=T(z)$.

Corollary 16. *For any real number $y \geqslant 1$, we have*

$$G(y)\log y = (n+1)T(y) + G(y)r(y)\log y,$$

where

$$|r(y)| \;\leqslant\; \left(\omega_{\mathbf{K}}(E) \;+\; \omega_{\mathbf{K}}(H) \;+\; 10n^2 \;+\; \frac{e^{75}|d_{\mathbf{K}}|^{1/3}(\log|d_{\mathbf{K}}|)^2}{\alpha_{\mathbf{K}}}\right)\frac{n}{\log y}.$$

Proof. Using Lemma 15 and adding $T(x,z)$ to both sides, we get

$$\begin{aligned} G(x,y)\log x &= (n+1)T(x,y) - nT\left(\frac{x}{y}, y\right) \\ &+ \mathrm{O}^*\left(n\left(\omega_{\mathbf{K}}(E) + \omega_{\mathbf{K}}(H) + 10n^2 + \frac{e^{75}|d_{\mathbf{K}}|^{1/3}(\log|d_{\mathbf{K}}|)^2}{\alpha_{\mathbf{K}}}\right)G(x,y)\right). \end{aligned}$$

Putting $x = y$, we get the corollary. □

From now onwards, for any real number $y > 3$, we denote by

(6)
$$U(y) = \log\left(\frac{n+1}{\log^{n+1} y}\,T(y)\right) \text{ and } L = n\left(\omega_{\mathbf{K}}(E) + \omega_{\mathbf{K}}(H) + 10n^2 + \frac{e^{75}|d_{\mathbf{K}}|^{1/3}(\log|d_{\mathbf{K}}|)^2}{\alpha_{\mathbf{K}}}\right).$$

Lemma 17. *For a real number z with $\log z \geqslant 3(n+1)L$, we have*

$$G(z) = c_{\mathbf{K},F}\log^n z\left(1 + \mathrm{O}^*\left(\frac{9(n+1)L}{\log z}\right)\right)$$

for some positive constant $c_{\mathbf{K},F}$ depending on $\mathbf{K}$ and F.

Proof. We first observe that for $\log z \geqslant 3(n+1)L$ and any real number $y \geqslant z$, we have

$$|U'(y)| = \left|-\frac{n+1}{y\log y} + \frac{T'(y)}{T(y)}\right| = \left|-\frac{n+1}{y\log y} + \frac{G(y)}{yT(y)}\right| = \left|\frac{r(y)}{1-r(y)}\frac{n+1}{y\log y}\right| \leqslant \frac{2(n+1)L}{y\log^2 y}.$$

This implies that the integral of $U'(y)$ from z to ∞ is convergent. Further

$$\left|-\int_z^\infty U'(y)dy\right| \;\leqslant\; \frac{2(n+1)L}{\log z} \;<\; 1.$$

Recall that

$$\frac{n+1}{\log^{n+1} z}T(z) = \exp(U(z)) = c_{\mathbf{K},F}\exp\left(-\int_z^\infty U'(y)dy\right)$$

for some constant $c_{\mathbf{K},F}$. We now observe that

$$\exp\left(-\int_z^\infty U'(y)dy\right) = 1 \;-\; \int_z^\infty U'(y)dy \;+\; \frac{1}{2!}\left(\int_z^\infty U'(y)dy\right)^2 - \cdots$$

and therefore

$$\exp\left(-\int_z^\infty U'(y)dy\right) = 1 + \mathrm{O}^*\left(\frac{2(n+1)L}{\log z} + \left(\frac{2(n+1)L}{\log z}\right)^2 + \cdots\right)$$
$$= 1 + \mathrm{O}^*\left(\frac{2(n+1)L}{\log z - 2(n+1)L}\right) = 1 + \mathrm{O}^*\left(\frac{6(n+1)L}{\log z}\right).$$

Further we have

$$\frac{1}{1-r(z)} = 1 + \frac{r(z)}{1-r(z)} = 1 + \mathrm{O}^*\left(\frac{L}{\log z - L}\right) = 1 + \mathrm{O}^*\left(\frac{2L}{\log z}\right)$$

since $\log z \geqslant 3L$. Applying Corollary 16 and combining the above, we get

$$G(z) = \frac{n+1}{(1-r(z))\log z}T(z) = c_{\mathbf{K},F}\log^n z\left(1+\mathrm{O}^*\left(\frac{2L}{\log z}\right)\right)\left(1+\mathrm{O}^*\left(\frac{6(n+1)L}{\log z}\right)\right)$$
$$= c_{\mathbf{K},F}\log^n z\left(1+\mathrm{O}^*\left(\frac{9(n+1)L}{\log z}\right)\right).$$

□

Remark 3.1. *If one wants a lower bound for $G(z)$ in the case $n=1$, one can use a simpler method that avoids relying on the sum $\rho(\mathfrak{p})(\log \mathfrak{N}\mathfrak{p})/\mathfrak{N}\mathfrak{p}$ as in Theorem 30 of* [4].

We conclude this section by computing the constant $c_{\mathbf{K},F}$.

Lemma 18. *We have*

$$c_{\mathbf{K},F} = \frac{\alpha_{\mathbf{K}}^n}{n!}\prod_{\mathfrak{N}\mathfrak{p}\leqslant n}\left(1-\frac{1}{\mathfrak{N}\mathfrak{p}}\right)^n\prod_{\mathfrak{p}\nmid H}(1+h(\mathfrak{p}))\left(1-\frac{1}{\mathfrak{N}\mathfrak{p}}\right)^n.$$

Proof. For a real parameter $s>0$, consider the series

$$M = \sum_{\substack{\partial\subseteq\mathcal{O}_{\mathbf{K}}\\ \partial\neq(0)\\ (\partial,H)=\mathcal{O}_{\mathbf{K}}}}\frac{h(\partial)}{\mathfrak{N}\partial^s}.$$

In the region $\Re s>0$, we have $M = \prod_{\mathfrak{p}\nmid H}\left(1+\frac{h(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^s}\right)$. Applying partial summation formula, we have

$$M = \lim_{x\to\infty}\left(\frac{\sum_{\substack{\mathfrak{N}\partial\leqslant x\\ (\partial,H)=\mathcal{O}_{\mathbf{K}}}}h(\partial)}{x^s} + s\int_1^x\frac{\sum_{\substack{\mathfrak{N}\partial\leqslant t\\ (\partial,H)=\mathcal{O}_{\mathbf{K}}}}h(\partial)}{t^{s+1}}dt\right) = \lim_{x\to\infty}\left(\frac{G(x)}{x^s} + s\int_1^x\frac{G(t)}{t^{s+1}}dt\right).$$

By Lemma 17, we have that $G(x)\ll\log^{n+1}x$ and hence $M = s\int_1^\infty\frac{G(t)}{t^{s+1}}dt$. We now split the integral into two parts. Let $z=3(n+1)L$, where L is as in (6). Then we have

$$M = s\int_1^z\frac{G(t)}{t^{s+1}}dt + s\int_z^\infty\frac{G(t)}{t^{s+1}}dt.$$

To estimate the first integral, we observe that for real $s > 0$, we have

$$s\int_1^z \frac{G(t)}{t^{s+1}}dt \;\leqslant\; s\int_1^z \frac{G(t)}{t}dt \;=\; sT(z).$$

Recall that

$$T(z) \;=\; \sum_{\substack{\mathfrak{N}\partial\leqslant z,\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} h(\partial)\log\frac{z}{\mathfrak{N}\partial} \;\leqslant\; \log z \sum_{\substack{\mathfrak{N}\partial\leqslant z\\ \partial|\mathcal{P}_{\mathbf{K}}(z)}} \frac{1}{f_1(\partial)} \;=\; \mathrm{O}(z\log z).$$

1 For the second integral, applying Lemma 17, we have

$$\begin{aligned} s\int_z^\infty \frac{G(t)}{t^{s+1}}dt \;&=\; s\int_z^\infty \frac{c_{\mathbf{K},F}\log^n t \;+\; \mathrm{O}\left(\log^{n-1} t\right)}{t^{s+1}}dt \\ &=\; s\int_1^\infty \frac{c_{\mathbf{K},F}\log^n t \;+\; \mathrm{O}\left(\log^{n-1} t\right)}{t^{s+1}}dt \;+\; \mathrm{O}(s\log^{n+1} z). \end{aligned}$$

We now use the fact that for $s > 0$,

$$\int_1^\infty \frac{\log^n t}{t^{s+1}}dt = \frac{\Gamma(n+1)}{s^{n+1}}.$$

Therefore

$$M \;=\; \prod_{\mathfrak{p}\nmid H}\left(1+\frac{h(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^s}\right) \;=\; c_{\mathbf{K},F}\frac{\Gamma(n+1)}{s^n} \;+\; \mathrm{O}\left(\frac{\Gamma(n)}{s^{n-1}}\right) \;+\; \mathrm{O}(s\log^{n+1} z + sz\log z).$$

It immediately follows that

$$c_{\mathbf{K},F} \;=\; \frac{1}{\Gamma(n+1)}\lim_{s\to 0^+} s^n\prod_{\mathfrak{p}\nmid H}\left(1+\frac{h(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^s}\right) = \frac{\alpha_{\mathbf{K}}^n}{n!}\prod_{\mathfrak{p}|H}\left(1-\frac{1}{\mathfrak{N}\mathfrak{p}}\right)^n \lim_{s\to 0^+}\prod_{\mathfrak{p}\nmid H}\left(1+\frac{h(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^s}\right)\left(1-\frac{1}{\mathfrak{N}\mathfrak{p}^{s+1}}\right)^n.$$

2 This completes the proof of the lemma. □

3 4. Proof of main theorem

Let z be a real number such that $z \geqslant 4$. We use $\mathfrak{N}(\alpha)$ to denote the absolute norm of the principal ideal (α) and $\mathcal{Q}$ to denote the set of all prime elements of $\mathcal{O}_{\mathbf{K}}$. Recall that

$$f_i(x) = a_i x + b_i \quad \text{for } 1 \leqslant i \leqslant n \qquad \text{and} \quad F(x) = \prod_{i=1}^n f_i(x).$$

4 We want to estimate

$$\begin{aligned} D = \sum_{\substack{\mathfrak{N}(\alpha)\leqslant u\\ f_i(\alpha)\in\mathcal{Q} \text{ for all } i}} 1 \;&\leqslant\; \sum_{\substack{\mathfrak{N}(\alpha)\leqslant z\\ f_i(\alpha)\in\mathcal{Q} \text{ for all } i}} 1 \;+\; \sum_{j=1}^n \sum_{\substack{\mathfrak{N}(f_j(\alpha))\leqslant z,\\ f_i(\alpha)\in\mathcal{Q} \text{ for all } i}} 1 \;+\; \sum_{\substack{z<\mathfrak{N}(\alpha)\leqslant u\\ (F(\alpha),\mathcal{P}_{\mathbf{K}}(z))=1}} 1 \\ &\leqslant\; \sum_{\substack{\mathfrak{N}(\alpha)\leqslant z\\ f_i(\alpha)\in\mathcal{Q} \text{ for all } i}} 1 \;+\; 2|\mu_{\mathbf{K}}|nz \;+\; \sum_{\substack{z<\mathfrak{N}(\alpha)\leqslant u,\\ (F(\alpha),\mathcal{P}_{\mathbf{K}}(z))=1}} 1, \end{aligned}$$

where $|\mu_{\mathbf{K}}|$ is the number of roots of unity in $\mathcal{O}_{\mathbf{K}}$. To estimate the first sum, we observe that for $u, v \in \mathcal{O}_{\mathbf{K}} \backslash \{0\}$, the norm of u, v are positive and

$$N_{\mathbf{K}/\mathbb{Q}}(u+v) = N_{\mathbf{K}/\mathbb{Q}}(u) + Tr_{\mathbf{K}/\mathbb{Q}}(u\bar{v}) + N_{\mathbf{K}/\mathbb{Q}}(v),$$

$\bar{v}$ denotes the complex conjugate of v. If $\mathcal{O}_{\mathbf{K}} = \mathbb{Z}[\sqrt{-d}]$ and $u\bar{v} = a + b\sqrt{-d}$, then

$$Tr_{\mathbf{K}/\mathbb{Q}}(u\bar{v}) = 2a \leqslant 2(a^2 + b^2 d) \leqslant 2N_{\mathbf{K}/\mathbb{Q}}(u\bar{v}).$$

Similarly if $\mathcal{O}_{\mathbf{K}} = \mathbb{Z}[\frac{1+\sqrt{-d}}{2}]$ and $u\bar{v} = a + \frac{b}{2} + \frac{b\sqrt{-d}}{2}$, we have

$$Tr_{\mathbf{K}/\mathbb{Q}}(u\bar{v}) = 2(a + \frac{b}{2}) \leqslant 2((a + \frac{b}{2})^2 + \frac{b^2 d}{4}) \leqslant 2N_{\mathbf{K}/\mathbb{Q}}(u\bar{v}).$$

Indeed, it is clearly true when $a + \frac{b}{2} \leqslant 0$ or $a + \frac{b}{2} \geqslant 1$. Now if $0 < a + \frac{b}{2} < 1$, then $a + \frac{b}{2} = \frac{1}{2}$ and $b \neq 0$ and therefore $1 \leqslant 2(\frac{1}{4} + \frac{b^2 d}{4})$. Thus in both cases $N_{\mathbf{K}/\mathbb{Q}}(u+v) \leqslant 4N_{\mathbf{K}/\mathbb{Q}}(uv)$. Therefore the first sum under consideration gives

$$\sum_{\substack{\mathfrak{N}(\alpha) \leqslant z, \\ f_i(\alpha) \in \mathcal{Q} \text{ for all } i}} 1 \;\leqslant \sum_{\substack{\mathfrak{N}(f_{i_0}(\alpha)) \leqslant 4\mathfrak{N}(a_{i_0} b_{i_0}) z, \\ f_i(\alpha) \in \mathcal{Q} \text{ for all } i}} 1 \;\leqslant\; 8|\mu_{\mathbf{K}}| \mathfrak{N}(a_{i_0} b_{i_0}) nz,$$

where $\mathfrak{N}(a_{i_0} b_{i_0}) = \min\{\mathfrak{N}(a_i b_i) \;:\; 1 \leqslant i \leqslant n\}$. Therefore

$$D \;\leqslant\; 10|\mu_{\mathbf{K}}| \mathfrak{N}(a_{i_0} b_{i_0}) nz \;+ \sum_{\substack{z < \mathfrak{N}(\alpha) \leqslant u, \\ (F(\alpha), \mathcal{P}_{\mathbf{K}}(z)) = 1}} 1.$$

Let us consider the sum

$$\sum_{\substack{\mathfrak{N}(\alpha) \leqslant u, \\ (F(\alpha), \mathcal{P}_{\mathbf{K}}(z)) = 1}} 1 \;=\; \sum_{\mathfrak{N}(\alpha) \leqslant u} \left(\sum_{\mathfrak{b} | (F(\alpha), \mathcal{P}_{\mathbf{K}}(z))} \mu(\mathfrak{b}) \right) \leqslant \sum_{\mathfrak{N}(\alpha) \leqslant u} \left(\sum_{\mathfrak{b} | (F(\alpha), \mathcal{P}_{\mathbf{K}}(z))} \lambda_{\mathfrak{b}} \right)^2.$$

Rearranging the sums, we get

$$\sum_{\mathfrak{N}(\alpha) \leqslant u} \left(\sum_{\mathfrak{b} | (F(\alpha), \mathcal{P}_{\mathbf{K}}(z))} \lambda_{\mathfrak{b}} \right)^2 = \sum_{\substack{\mathfrak{b}_1, \mathfrak{b}_2 | \mathcal{P}_{\mathbf{K}}(z), \\ \mathfrak{N}\mathfrak{b}_i \leqslant z}} \lambda_{\mathfrak{b}_1} \lambda_{\mathfrak{b}_2} \sum_{\substack{\mathfrak{N}(\alpha) \leqslant u, \\ [\mathfrak{b}_1, \mathfrak{b}_2] | F(\alpha)}} 1.$$

Let $\mathfrak{b} = [\mathfrak{b}_1, \mathfrak{b}_2]$. To estimate the inner sum, we need to count $\alpha \in \mathcal{O}_{\mathbf{K}}$ such that α lies in one of the $\rho(\mathfrak{b})$ classes in $\mathcal{O}_{\mathbf{K}}/\mathfrak{b}$. If $\mathfrak{b} \mid \mathcal{P}_{\mathbf{K}}(z)$ and $\mathfrak{b}_0$ is the largest divisor of $\mathfrak{b}$ which is co-prime to $E = \prod_{i=1}^n a_i \prod_{1 \leqslant i < j \leqslant n} (a_i b_j - a_j b_i)$, we can write $\rho(\mathfrak{b}) = n^{\omega(\mathfrak{b}_0)} \rho(\frac{\mathfrak{b}}{\mathfrak{b}_0})$. Applying Theorem 3 for $\mathfrak{a} = \mathcal{O}_{\mathbf{K}}$, $\mathfrak{q} = \mathfrak{b}$, we get for $z \leqslant \sqrt{u}$

(7)
$$\sum_{\substack{\mathfrak{b}_1, \mathfrak{b}_2 | \mathcal{P}_{\mathbf{K}}(z) \\ \mathfrak{N}\mathfrak{b}_i \leqslant z}} \lambda_{\mathfrak{b}_1} \lambda_{\mathfrak{b}_2} \sum_{\substack{\mathfrak{N}(\alpha) \leqslant u, \\ [\mathfrak{b}_1, \mathfrak{b}_2] | F(\alpha)}} 1$$
$$= \sum_{\substack{\mathfrak{b}_1, \mathfrak{b}_2 | \mathcal{P}_{\mathbf{K}}(z) \\ \mathfrak{N}\mathfrak{b}_i \leqslant z}} \lambda_{\mathfrak{b}_1} \lambda_{\mathfrak{b}_2} \left(\frac{c_{\mathbf{K}} \rho([\mathfrak{b}_1, \mathfrak{b}_2]) u}{\mathfrak{N}[\mathfrak{b}_1, \mathfrak{b}_2]} + \mathrm{O}^* \left(10^{14} \rho([\mathfrak{b}_1, \mathfrak{b}_2]) \sqrt{\frac{u}{\mathfrak{N}[\mathfrak{b}_1, \mathfrak{b}_2]}} \right) \right),$$

where $c_{\mathbf{K}} = \frac{2\pi}{\sqrt{|d_{\mathbf{K}}|}}$. Note that the main term is $\sum_{\mathfrak{b}_1,\mathfrak{b}_2|\mathcal{P}_{\mathbf{K}}(z)} \frac{\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}}{f([\mathfrak{b}_1,\mathfrak{b}_2])}$, where f is as defined in (4). Hence

$$\sum_{\mathfrak{b}_1,\mathfrak{b}_2|\mathcal{P}_{\mathbf{K}}(z)} \frac{\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2} f((\mathfrak{b}_1,\mathfrak{b}_2))}{f(\mathfrak{b}_1)f(\mathfrak{b}_2)} = \sum_{\mathfrak{b}_1,\mathfrak{b}_2|\mathcal{P}_{\mathbf{K}}(z)} \frac{\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}}{f(\mathfrak{b}_1)f(\mathfrak{b}_2)} \sum_{\mathfrak{a}|(\mathfrak{b}_1,\mathfrak{b}_2)} f_1(\mathfrak{a}) = \sum_{\mathfrak{a}|\mathcal{P}_{\mathbf{K}}(z)} f_1(\mathfrak{a}) \left(\sum_{\substack{\mathfrak{a}|\mathfrak{c},\\ \mathfrak{c}|\mathcal{P}_{\mathbf{K}}(z)}} \frac{\lambda_{\mathfrak{c}}}{f(\mathfrak{c})} \right)^2.$$

Further, we observe that

$$\sum_{\substack{\mathfrak{c}|\mathcal{P}_{\mathbf{K}}(z)\\ \mathfrak{a}|\mathfrak{c}}} \frac{\lambda_{\mathfrak{c}}}{f(\mathfrak{c})} = G(z)^{-1} \sum_{\substack{\mathfrak{c}|\mathcal{P}_{\mathbf{K}}(z)\\ \mathfrak{a}|\mathfrak{c}}} \frac{\mu(\mathfrak{c})}{f_1(\mathfrak{c})} \sum_{\substack{\mathfrak{N}(\mathfrak{g})\leqslant \frac{z}{\mathfrak{N}(\mathfrak{c})}\\ (\mathfrak{g},\mathfrak{c}H)=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{g})}{f_1(\mathfrak{g})}.$$

2 Writing $\mathfrak{c} = \mathfrak{h}\mathfrak{a}$ with $(\mathfrak{h},\mathfrak{a}) = \mathcal{O}_{\mathbf{K}}$, we get

$$\frac{\mu(\mathfrak{a})}{f_1(\mathfrak{a})} G(z)^{-1} \sum_{\substack{\mathfrak{N}(\mathfrak{h})\leqslant \frac{z}{\mathfrak{N}(\mathfrak{a})},\\ \mathfrak{h}|\mathcal{P}_{\mathbf{K}}(z),\\ (\mathfrak{h},\mathfrak{a})=\mathcal{O}_{\mathbf{K}}}} \frac{\mu(\mathfrak{h})}{f_1(\mathfrak{h})} \sum_{\substack{\mathfrak{N}(\mathfrak{g})\leqslant \frac{z}{\mathfrak{N}(\mathfrak{h}\mathfrak{a})}\\ (\mathfrak{g},\mathfrak{h}\mathfrak{a}H)=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{g})}{f_1(\mathfrak{g})} = \frac{\mu(\mathfrak{a})}{f_1(\mathfrak{a})} G(z)^{-1} \sum_{\substack{\mathfrak{N}(\mathfrak{h})\leqslant \frac{z}{\mathfrak{N}(\mathfrak{a})},\\ \mathfrak{h}|\mathcal{P}_{\mathbf{K}}(z),\\ (\mathfrak{h},\mathfrak{a})=\mathcal{O}_{\mathbf{K}}}} \sum_{\substack{\mathfrak{N}(\mathfrak{g})\leqslant \frac{z}{\mathfrak{N}(\mathfrak{h}\mathfrak{a})}\\ (\mathfrak{g},\mathfrak{h}\mathfrak{a}H)=\mathcal{O}_{\mathbf{K}}}} \mu(\mathfrak{h}) \frac{\mu^2(\mathfrak{g}\mathfrak{h})}{f_1(\mathfrak{g}\mathfrak{h})}.$$

Setting $\mathfrak{a}_1 = \mathfrak{g}\mathfrak{h}$ gives

$$\sum_{\substack{\mathfrak{c}|\mathcal{P}_{\mathbf{K}}(z)\\ \mathfrak{a}|\mathfrak{c}}} \frac{\lambda_{\mathfrak{c}}}{f(\mathfrak{c})} = \frac{\mu(\mathfrak{a})}{f_1(\mathfrak{a})} G(z)^{-1} \sum_{\substack{\mathfrak{N}(\mathfrak{a}_1)\leqslant \frac{z}{\mathfrak{N}(\mathfrak{a})}\\ (\mathfrak{a}_1,\mathfrak{a}H)=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{a}_1)}{f_1(\mathfrak{a}_1)} \sum_{\mathfrak{h}|\mathfrak{a}_1} \mu(\mathfrak{h}) = G(z)^{-1} \frac{\mu(\mathfrak{a})}{f_1(\mathfrak{a})}.$$

Therefore the main term in (7) is $c_{\mathbf{K}} u G(z)^{-1}$. Applying Lemma 17 and Lemma 18, we have for $\log z \geqslant 18(n+1)L$

$$G(z) = \frac{\alpha_{\mathbf{K}}^n}{n!} \prod_{\mathfrak{p}|H} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^n \prod_{\mathfrak{p}\nmid H} (1 + h(\mathfrak{p})) \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^n \log^n z \left(1 + \mathrm{O}^*\left(\frac{9(n+1)L}{\log z}\right)\right).$$

3 To deal with the error term, we use Lemma 12. Combining everything, we get for $z \leqslant \sqrt{u}$

$$D \leqslant c_{\mathbf{K}} u G(z)^{-1} + 10|\mu_{\mathbf{K}}| \mathfrak{N}(a_{i_0} b_{i_0}) n z + 10^{14} (3n)^{4\pi_{\mathbf{K}}(2n)} \zeta_{\mathbf{K}} \left(\frac{3}{2}\right)^{8n} z\sqrt{u}.$$

4 We now simplify the above expression to get

$$D \leqslant \frac{2\pi}{\sqrt{|d_{\mathbf{K}}|}} u G(z)^{-1} + 3^{61n} n^{16n} \mathfrak{N}(a_{i_0} b_{i_0}) z\sqrt{u}.$$

Therefore, if we choose

$$z = \frac{\pi\sqrt{u} G(u)^{-1}}{2\sqrt{|d_{\mathbf{K}}|} 3^{61n} n^{16n} \mathfrak{N}(a_{i_0} b_{i_0})} \leqslant \frac{\pi\sqrt{u} G(z)^{-1}}{2\sqrt{|d_{\mathbf{K}}|} 3^{61n} n^{16n} \mathfrak{N}(a_{i_0} b_{i_0})}.$$

5 For $\log z \geqslant 18(n+1)L$, we have

$$D \leqslant \frac{5}{4} \cdot \frac{2\pi}{\sqrt{|d_{\mathbf{K}}|}} G(z)^{-1} u.$$

We now compute the lower bound for u. Note that

$$(4n)^n \frac{\sqrt{u}}{\log^n u} \;\geqslant\; u^{\frac{1}{4}}.$$

Therefore for

$$u^{\frac{1}{4}} \;\geqslant\; \frac{(4n)^n \sqrt{|d_{\mathbf{K}}|} 3^{62n} n^{16n} \mathfrak{N}(a_{i_0} b_{i_0}) \alpha_{\mathbf{K}}^n \prod_{\mathfrak{p}|H} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^n \prod_{\mathfrak{p}\nmid H} (1 + h(\mathfrak{p})) \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^n}{n!\, \pi \exp(-18(n+1)L)},$$

6 we have

$$D \;\leqslant\; \frac{5 \left(\prod_{\mathfrak{p}|H} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-n} \prod_{\mathfrak{p}\nmid H} (1 + h(\mathfrak{p}))^{-1} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-n}\right) n! |\mu_{\mathbf{K}}| u}{2\alpha_{\mathbf{K}}^{n-1} h_{\mathbf{K}} \log^n \frac{\pi \sqrt{u} G(u)^{-1}}{2\sqrt{|d_{\mathbf{K}}|} 3^{61n} n^{16n} \mathfrak{N}(a_{i_0} b_{i_0})}}.$$

We now consider the product

$$\prod_{\mathfrak{p}\nmid H} (1+h(\mathfrak{p})) \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^n = \prod_{\mathfrak{p}\nmid H} \left(1 + \frac{\rho(\mathfrak{p})}{\mathfrak{N}\mathfrak{p} - \rho(\mathfrak{p})}\right) \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^n \leqslant \prod_{\mathfrak{p}\nmid H} \left(\left(1 + \frac{1}{\mathfrak{N}\mathfrak{p} - \rho(\mathfrak{p})}\right) \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)\right)^n.$$

Further we have

$$\left(1 + \frac{1}{\mathfrak{N}\mathfrak{p} - \rho(\mathfrak{p})}\right) \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right) = \left(1 + \frac{\rho(\mathfrak{p}) - 1}{\mathfrak{N}\mathfrak{p}(\mathfrak{N}\mathfrak{p} - \rho(\mathfrak{p}))}\right) \leqslant \left(1 + \frac{1}{\mathfrak{N}\mathfrak{p}(\mathfrak{N}\mathfrak{p} - \rho(\mathfrak{p}))}\right)^{n-1}.$$

This gives

$$\prod_{\mathfrak{p}\nmid H} (1 + h(\mathfrak{p})) \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^n \;\leqslant\; \prod_{\mathfrak{p}\nmid H} \left(1 + \frac{1}{\mathfrak{N}\mathfrak{p}(\mathfrak{N}\mathfrak{p} - \rho(\mathfrak{p}))}\right)^{n(n-1)}.$$

Finally

$$\prod_{\mathfrak{p}\nmid H} \left(1 + \frac{1}{\mathfrak{N}\mathfrak{p}(\mathfrak{N}\mathfrak{p} - \rho(\mathfrak{p}))}\right)^{n(n-1)} \;\leqslant\; \prod_{\substack{\mathfrak{N}\mathfrak{p} < 2n \\ \mathfrak{p}\nmid H}} \left(1 + \frac{1}{\mathfrak{N}\mathfrak{p}(\mathfrak{N}\mathfrak{p} - \rho(\mathfrak{p}))}\right)^{n(n-1)} \prod_{\mathfrak{p}} \left(1 + \frac{1}{\mathfrak{N}\mathfrak{p}^2}\right)^{2n(n-1)}.$$

Therefore the constant

$$\prod_{\mathfrak{p}\nmid H} (1 + h(\mathfrak{p})) \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^n \leqslant 2^{n(n-1)\pi_{\mathbf{K}}(2n)} \zeta_{\mathbf{K}}(2)^{2n(n-1)} \leqslant 2^{6n^3}.$$

Thus for

$$u^{\frac{1}{4}} \;\geqslant\; \exp(18(n+1)L) \left(\frac{\sqrt{|d_{\mathbf{K}}|} 3^{23n^3} n^{17n} \mathfrak{N}(a_{i_0} b_{i_0}) \alpha_{\mathbf{K}}^n}{n! \pi}\right),$$

7 we have

$$D \;\leqslant\; \frac{5 \left(\prod_{\mathfrak{p}|H} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-n} \prod_{\mathfrak{p}\nmid H} (1 + h(\mathfrak{p}))^{-1} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-n}\right) n! |\mu_{\mathbf{K}}| u}{2\alpha_{\mathbf{K}}^{n-1} h_{\mathbf{K}} \log^n \frac{n! \pi \sqrt{u}}{\sqrt{|d_{\mathbf{K}}|} 3^{22n^3} n^{16n} \mathfrak{N}(a_{i_0} b_{i_0}) \alpha_{\mathbf{K}}^n \log^n u}}.$$

Further since $u^{\frac{1}{4n}} > \log u^{\frac{1}{4n}}$, we get for

$$u^{\frac{1}{4}} \geqslant \exp(18(n+1)L)\left(\frac{\sqrt{|d_{\mathbf{K}}|}\,3^{23n^3}n^{17n}\mathfrak{N}(a_{i_0}b_{i_0})\alpha_{\mathbf{K}}^n}{n!\pi}\right),$$

we have

$$D \leqslant \frac{5\left(\prod_{\mathfrak{p}|H}\left(1-\frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-n}\prod_{\mathfrak{p}\nmid H}(1+h(\mathfrak{p}))^{-1}\left(1-\frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-n}\right)n!|\mu_{\mathbf{K}}|u}{2\alpha_{\mathbf{K}}^{n-1}h_{\mathbf{K}}\log^n\frac{n!\pi\ u^{\frac{1}{4}}}{\sqrt{|d_{\mathbf{K}}|}\,3^{23n^3}n^{17n}\mathfrak{N}(a_{i_0}b_{i_0})\alpha_{\mathbf{K}}^n}}.$$

Note that by relabelling a_i's and b_i's for $1 \leqslant i \leqslant n$, we can choose i_0 to be equal to 1.

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